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Low Reynolds number oscillatory flow through a hole in a wall

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For a viscous incompressible fluid the low Reynolds number description of the flow generated by a three-dimensional point source of oscillating strength situated on a wall is approximately quasi-steady in the neighbourhood of the singularity. This quasi-steady solution contains a number of unacceptable features, the principal one being that it is not a uniformly valid approximation within a small region surrounding the source point. In addition to this the vorticity in this region is predicted to be zero everywhere on the wall except at the singular point where it is infinite, which does not seem to be a physically reasonable distribution. When account is taken of the finite radius of the hole through which the fluid is driven and the finite width of the wall, the above difficulties are resolved yielding results that are quite realistic and informative.

1. Introduction

Recent work by Tuck (1970) investigates the viscous flow at low Reynolds number associated with a two-dimensional point mass source of oscillating strength situated on a wall. The results indicate that the dominant part of the flow in the neighbourhood of the source point is quasi-steady in the sense that the asymptotic solution predicted for this region is the product of the corresponding steady solution and a sinusoidal time component. A similar situation exists for the analogous oscillatory axisymmetric problem of a three-dimensional point source on a wall, but for this case the three-dimensional quasi-steady solution is not uniformly valid in the region surrounding the source point (unlike the two-dimensional result) since it predicts that the convective terms in the governing non-linear non-dimensional stream function equation become just as important as the diffusive terms, as the source point is approached.

The difficulty is removed if the problem is attempted with the point source replaced by a circular hole of finite width through which the fluid flows with the same oscillatory volume flux. This problem is formulated in §2 and the analysis of the flow field at low Reynolds number is performed by considering an inner region with the length scale chosen as the radius of the hole and an adjacent outer region. The dominant part of the outer region solution is due to a point source of oscillating strength on the wall, the relevant results obtained in §4 resembling those obtained by Tuck in the two-dimensional case.

The inner region geometry is suitable for the problem to be treated in axisymmetric oblate spheroidal co-ordinates, this being carried out in §3. The quasi-steady results obtained are quite interesting in that they are self-consistent in the immediate neighbourhood of the hole, unlike the corresponding point source results. For the special problem of steady flow (zero frequency of oscillation) through a finite hole, the results are uniformly valid throughout the whole flow field for small values of the Reynolds number, a conclusion that does not appear to have been previously drawn.

Although taking account of the finite width of the hole removes some unacceptable features of the point source analysis (steady or unsteady), there still appears to be predicted an unusual vorticity distribution on the wall. It is shown, however, that consideration of the finite thickness of the wall eliminates this further abnormal feature, just as consideration of the finite width of the hole enabled the nature of the flow in the hole's vicinity to be resolved.

2. Formulation of the problem

Consider an infinite impervious rigid wall containing a circular hole of radius a. An incompressible viscous fluid is flowing backward and forward through the hole at a rate such that the volume flux at any instant is $2\pi m \cos \sigma t_*$ where t_* denotes the dimensional time. If $\mathbf{x}_*$ denotes the dimensional position vector referred to the centre of the hole as the origin, and Ψ_* denotes the dimensional stream function, corresponding non-dimensional quantities t, $\mathbf{x}$ and Ψ are defined by

$$\mathbf{x}_* = a\mathbf{x}, \quad t_* = \sigma^{-1}t, \quad \Psi_* = m\Psi.$$

The geometry of the problem suggests the use of non-dimensional axisymmetric oblate spheroidal co-ordinates (Y, Z, ϕ) related to the cylindrical polar co-ordinates (ρ, ϕ, z) by

$$\rho = [(1+Y^2)(1-Z^2)]^{\frac{1}{2}}, \quad z = YZ,$$

the fluid and the wall occupying the region $0 \leqslant Y < \infty$, $-1 \leqslant Z \leqslant 1$, $0 \leqslant \phi \leqslant 2\pi$. The surfaces $Y =$ constant are oblate ellipsoids of revolution about the z axis while the surfaces $Z =$ constant are one-sheeted hyperboloids of revolution about the z axis. In particular the axis of symmetry is represented by $Z = \pm 1$, the wall is represented by $Z = 0$ and the hole through which the fluid pulsates is represented by $Y = 0$.

In terms of the above non-dimensional variables the axisymmetric, swirl-free equation for the stream function (in the absence of body forces and thermal conduction) is

$$D^4\Psi - S\frac{\partial}{\partial t}(D^2\Psi) = \frac{R}{Y^2+Z^2}\left\{\frac{\partial(\Psi, D^2\Psi)}{\partial(Y,Z)} + 2D^2\Psi L\Psi\right\}, \tag{2.1}$$

where
$$D^2 \equiv \left(\frac{1+Y^2}{Y^2+Z^2}\right)\frac{\partial^2}{\partial Y^2} + \left(\frac{1-Z^2}{Y^2+Z^2}\right)\frac{\partial^2}{\partial Z^2},$$

$$L \equiv \left(\frac{Z}{1-Z^2}\right)\frac{\partial}{\partial Y}+\left(\frac{Y}{1+Y^2}\right)\frac{\partial}{\partial Z},$$

the Reynolds number R is defined by

$$R = m/a\nu$$

(ν denotes the kinematic viscosity) and the parameter S is defined by

$$S = a^2\sigma/\nu,$$

the ratio of the Reynolds number to the Strouhal number.

The respective boundary conditions are:

(i) the no-slip conditions

$$\Psi = \partial\Psi/\partial Z = 0 \quad \text{on} \quad Z = 0;$$

(ii) the condition

$$\Psi = \cos t \quad \text{on} \quad Z = \pm 1,$$

which produces the correct flux through the hole;

(iii) the condition

$$[1/(1-Z^2)^{\frac{1}{2}}]\,\partial\Psi/\partial Y = 0 \quad \text{on} \quad Z = \pm 1$$

to ensure zero transverse velocity on the axis of symmetry;

(iv) the requirement that the velocity components must tend to zero as $|\mathbf{x}| \to \infty$.

3. The inner region

For the case $R \ll 1$, $S \ll 1$ an approximate low Reynolds number expression for the stream function is given by

$$\Psi(Y,Z,t) = \Psi^{(0)}(Y,Z,t) + (\text{terms of higher order in } R \text{ or } S).$$

If this expression is substituted into (2.1) it is seen that $\Psi^{(0)}$ satisfies the zeroth approximate equation

$$D^4\Psi^{(0)} = 0$$

under the same boundary conditions (i)–(iv) as for Ψ. With

$$\Psi^{(0)}(Y,Z,t) \equiv \psi^{(0)}(Y,Z)\cos t,$$

the problem reduces to solving

$$D^4\psi^{(0)} = 0,$$

subject to the same boundary conditions as apply to Ψ except that (ii) becomes $\psi^{(0)} = 1$ on $Z = \pm 1$. The reduced problem then corresponds to that for steady flow through a hole, which has been solved previously by Sampson (1891) using separation of variables and recursive properties of integrals of Legendre functions. It is productive and simpler to consider the more general equation

$$L_k^2\psi^{(0)} = 0,$$

where

$$L_k \equiv \frac{1}{Y^2+Z^2}\left\{(1+Y^2)\frac{\partial^2}{\partial Y^2}+(1-Z^2)\frac{\partial^2}{\partial Z^2}+(1-k)\,Y\frac{\partial}{\partial Y}-(1-k)\,Z\frac{\partial}{\partial Z}\right\},$$

with boundary conditions

$$\begin{aligned} \psi^{(0)} &= 0 \quad \text{on} \quad Z = 0, \\ \partial\psi^{(0)}/\partial Z &= 0 \quad \text{on} \quad Z = 0, \\ \psi^{(0)} &= 1 \quad \text{on} \quad Z = \pm 1, \\ (1-Z^2)^{-\frac{1}{2}k}\partial\psi^{(0)}/\partial Y &= 0 \quad \text{on} \quad Z = \pm 1, \end{aligned}$$

and the appropriate far-field conditions.

The form of the boundary conditions suggests seeking a solution that is independent of Y, and a generalized axisymmetric potential theorem due to Weinstein (1955) yields

$$\psi^{(0)} = K_1\int^Z (1-\xi^2)^{\frac{1}{2}(k-1)}d\xi + K_2\int^Z (1-\xi^2)^{\frac{1}{2}(k+1)}d\xi,$$

where K_1 and K_2 are arbitrary constants. The first integral is a solution of $L_k\psi^{(0)} = 0$, so the vorticity arises solely from the second integral.

The equation and boundary conditions associated with the two-dimensional problem of low Reynolds number flow through an infinite slit correspond to the above with $k = 0$, from which the solution is seen to be

$$\psi^{(0)} = \begin{cases} (2/\pi)\,(\arcsin Z - Z[1-Z^2]^{\frac{1}{2}}) & (0 \leqslant Z \leqslant 1), \\ -(2/\pi)\,(\arcsin Z - Z[1-Z^2]^{\frac{1}{2}}) & (-1 \leqslant Z \leqslant 0), \end{cases}$$

where the principal value of the inverse sine function is to be used. This result agrees with the steady two-dimensional result obtained by Green (1944). In much the same way as is to be shown for the three-dimensional situation, the two-dimensional inner unsteady result can be considered in conjunction with the outer solution already obtained by Tuck enabling the point source description and its inherent prediction of infinite velocity to be replaced by a more suitable flow geometry.

For $k = 1$ the equation and boundary conditions are those associated with the inner region of the axisymmetric problem being considered and yield the important result

$$\Psi^{(0)} = \begin{cases} Z^3\cos t & (0 \leqslant Z \leqslant 1), \\ -Z^3\cos t & (-1 \leqslant Z \leqslant 0), \end{cases} \tag{3.1}$$

from which the corresponding zeroth-order approximate velocity components are $(\mp 3Z^2\cos t/[(1+Y^2)(Y^2+Z^2)]^{\frac{1}{2}}, 0, 0)$. The positive and negative signs associated with these expressions for the stream function and velocity components indicate that the flow on one side of the wall is into the hole while on the other side it is away from the hole. These results predict that in the inner region the zeroth-order approximate streamlines are hyperbolae, that the velocity is nowhere infinite and that the velocity in the hole ($Y = 0$) is parallel to the axis of symmetry with a non-dimensional value varying monotonically from $-3\cos t$ at the centre to zero at the edge (a type of axisymmetric oscillatory shear flow).

The vorticity derived from (3.1) has components

$$(0, 0, \mp 6Z(1-Z^2)^{\frac{1}{2}}\cos t/\{(Y^2+Z^2)\,[1+Y^2]^{\frac{1}{2}}\}).$$

This was not considered by Sampson but is interesting in that it predicts zero vorticity not only on the axis of symmetry but also on the wall, except at the edge of the hole ($Y = 0$, $Z = 0$) where the vorticity becomes infinite. For the semi-infinite region $0 \leqslant Z \leqslant 1$ on one side of the wall, it is seen that on any ellipsoid $Y = Y_0$, at a fixed time t_0, there is a maximum of negative or positive vorticity at $Z = Y_0/[2(1+Y_0^2)]^{\frac{1}{2}}$ with a value

$$-6[2+Y_0^2]^{\frac{1}{2}}\cos t_0/\{Y_0(3+2Y_0^2)[1+Y_0^2]^{\frac{1}{2}}\}.$$

(The sign of this vorticity will be opposite to the sign of $\cos t_0$.) The locus of these 'maxima' is the curve $Y^2 = 2Z^2(1+Y^2)$ $(0 \leqslant Z \leqslant 1,\ 0 \leqslant Y < \infty)$ which is asymptotic to $\theta = \frac{1}{4}\pi$ as $Y \to \infty$. (The line $\theta = \frac{1}{4}\pi$ is the corresponding locus for the steady point source problem.) On each hyperboloid the amount of positive or negative vorticity approaches zero monotonically as $Y \to \infty$, and so it is concluded that all the vorticity is created at the edge of the hole and diffused out into the fluid.

The physically unreasonable vorticity distribution on the wall is avoided by taking account of the finite width of the wall so that the no-slip boundary conditions are imposed on the thin hyperboloid $Z = \epsilon$ (where ϵ is small) instead of on $Z = 0$. The zeroth-order approximate solution of this problem is

$$\Psi^{(0)} = \frac{Z^3 - 3\epsilon^2 Z + 2\epsilon^3}{(1-\epsilon)^2(1+2\epsilon)}\cos t$$

and the non-zero vorticity component is

$$-6Z[1-Z^2]^{\frac{1}{2}}\cos t/\{(1-\epsilon)^2(1+2\epsilon)(Y^2+Z^2)[1+Y^2]^{\frac{1}{2}}\}.$$

Thus on the wall $Z = \epsilon$ the vorticity distribution is approximated by the expression $-6\epsilon[1-\epsilon^2]^{\frac{1}{2}}\cos t/\{(1-\epsilon)^2(1+2\epsilon)(Y^2+\epsilon^2)[1+Y^2]^{\frac{1}{2}}\}$ which is large (but finite) at $Y = 0$, is non-zero for other finite Y, and tends to zero in the far part of the inner region as $Y \to \infty$.

The self-consistency of solution (3.1) is examined by considering the various terms in equation (2.1). A typical viscous term has the form

$$Z(1-Z^2)(1+Y^2)\cos t/(Y^2+Z^2)^3,$$

while of the neglected terms a typical non-linear term has the form

$$RZ^3Y(1-Z^2)\cos^2 t/(Y^2+Z^2)^3$$

and the local inertia term has the form $SZ(1-Z^2)\sin t/(Y^2+Z^2)$.

As $Y \to 0$ the ratio of the non-linear terms to the viscous terms is $O(RY)$ and the ratio of the local inertia terms to the viscous terms is $O(S)$, which indicates that the solution is self-consistent in the neighbourhood of the hole for small values of R and S. This analysis includes the case of steady flow which is obtained by taking $\sigma = S = 0$; then the relevant steady flow results are those already given in this section with the time factor omitted.

As $Y \to \infty$ it is useful to consider the steady and unsteady cases separately. For the steady case the ratio of the non-linear terms to the viscous terms is $O(R/Y)$ as $Y \to \infty$, hence the low Reynolds number steady solution

$$\psi^{(0)} = \left\{\begin{matrix} Z^3 & (0 \leqslant Z \leqslant 1) \\ -Z^3 & (-1 \leqslant Z < 0) \end{matrix}\right\} \tag{3.2}$$

is a uniformly valid approximation over the entire region occupied by the fluid. Since $Z \sim \cos\theta$ and $Y \sim r$ as $Y \to \infty$, the solution in the far field approaches $\pm\cos^3\theta$, which is the classical viscous solution for the stream function in the low Reynolds number problem involving a steady three-dimensional point source (or sink) on a wall—a solution which is invalid near the source point.

The above analysis shows that this point source solution has been superseded by the uniformly valid approximate solution (3.2) which takes account of the finite width of the hole through which the fluid flows.

For the case of unsteady flow at low Reynolds numbers the ratio of the non-linear terms to the viscous terms is $O(R/Y)$ as $Y \to \infty$, but the ratio of the local inertia terms to the viscous terms has to be considered as well. This ratio is $O(SY^2)$ as $Y \to \infty$, which suggests that the local inertia terms should be included in an outer region analysis when Y is $O(S^{-\frac{1}{2}})$. As this region is approached, the solution (3.1) tends towards that for a quasi-steady viscous flow due to a point source of oscillating strength situated on the wall at the origin.

4. The outer region

In order to take account of the importance of local inertia effects at large distances from the hole, the problem formulated in §2 is reconsidered by introducing new variables $\mathbf{x}' = S^{\frac{1}{2}}\mathbf{x}$ and $\Psi'(\mathbf{x}', t) = \Psi(\mathbf{x}, t)$. For $R \ll 1, S \ll 1$ a suitable expansion in this outer region is

$$\Psi'(\rho', z', t) = \Psi^{(0)\prime}(\rho', z', t) + (\text{terms of higher order in } R \text{ or } S),$$

where (ρ', ϕ, z') are cylindrical polar co-ordinates referred to the centre of the hole as origin. Substitution of this expansion into (2.1) yields the zeroth-order approximate equation

$$\mathscr{D}'^4\Psi^{(0)\prime} - \partial/\partial t\{\mathscr{D}'^2\Psi^{(0)\prime}\} = 0, \tag{4.1}$$

where

$$\mathscr{D}'^2 \equiv \frac{\partial^2}{\partial \rho'^2} - \frac{1}{\rho'}\frac{\partial}{\partial \rho'} + \frac{\partial^2}{\partial z'^2}.$$

In terms of the new variables it is seen from the boundary conditions (i)–(iv) of §2 that

$$\Psi^{(0)\prime} = \partial\Psi^{(0)\prime}/\partial z' \mp 0 \quad \text{on} \quad z' = 0(\rho' \mp 0),$$

$$\Psi^{(0)\prime} = \cos t \quad \text{on} \quad \rho' = 0,$$

$$\partial\Psi^{(0)\prime}/\partial z' = 0 \quad \text{on} \quad \rho' = 0,$$

with a zero velocity far-field condition. This outer problem for $\Psi^{(0)\prime}$ is the axi-symmetric equivalent of Tuck's problem and a very similar method is used to solve it.

Three simplifications are made at this stage; first, the primes on the variables and operators are omitted as no confusion arises in the remainder of the analysis, second, the symmetry of the flow about the wall except for the flow direction makes it sufficient to consider the fluid on one side of the wall only, and third, the boundary conditions suggest that the linear equation (4.1) has a solution of the form

$$\Psi^{(0)}(\rho, z, t) \equiv \mathrm{Re}\{\psi^{(0)}(\rho, z)\, e^{it}\}.$$

This means that the complex function $\psi^{(0)}(\rho, z)$ satisfies

$$\mathscr{D}^4\psi^{(0)} - \alpha^2\mathscr{D}^2\psi^{(0)} = 0,$$

where $\alpha^2 = i$, with boundary conditions

$$\psi^{(0)} = \partial\psi^{(0)}/\partial z = 0 \quad \text{on} \quad z = 0(\rho \neq 0),$$

$$\psi^{(0)} = 1,\ \partial\psi^{(0)}/\partial z = 0 \quad \text{on} \quad \rho = 0(z \geqslant 0)$$

and the zero velocity far-field condition.

To solve this differential equation set

$$\Omega^{(0)} = -\mathscr{D}^2\psi^{(0)}, \tag{4.2}$$

which reduces the fourth-order differential equation to a second-order one, namely

$$\mathscr{D}^2\Omega^{(0)} - \alpha^2\Omega^{(0)} = 0.$$

If separation of variables is applied, the axis and far-field conditions yield

$$\Omega^{(0)} = \int_0^\infty A(\lambda)\,\rho J_1(\lambda\rho)\,e^{-\beta z}d\lambda,$$

where $\lambda^2 = \beta^2 - \alpha^2$ and the real part of β is positive. When this result is substituted into (4.2) a similar technique gives

$$\psi^{(0)} = 1 + \int_0^\infty \rho J_1(\lambda\rho)\left\{B(\lambda)\,e^{-\lambda z} - \frac{A(\lambda)}{\alpha^2}\,e^{-\beta z}\right\}d\lambda.$$

The boundary conditions on the wall ($z = 0$) yield two integral equations

$$\int_0^\infty \rho J_1(\lambda\rho)\left\{\frac{\beta A}{\alpha^2} - \lambda B\right\}d\lambda = 0$$

and

$$1 + \int_0^\infty \rho J_1(\lambda\rho)\left\{B - \frac{A}{\alpha^2}\right\}d\lambda = 0,$$

which can be solved by inverse Hankel transforms from which it can be deduced that

$$A(\lambda) = -\lambda(\lambda+\beta), \quad B(\lambda) = \beta/(\lambda-\beta),$$

and hence

$$\Omega^{(0)}(\rho, z) = -\rho\int_0^\infty \lambda(\lambda+\beta)\,e^{-\beta z}J_1(\lambda\rho)\,d\lambda,$$

$$\psi^{(0)}(\rho, z) = 1 + \frac{\rho}{\alpha^2}\int_0^\infty (\lambda+\beta)(\lambda e^{-\beta z} - \beta e^{-\lambda z})\,J_1(\lambda\rho)\,d\lambda$$

$$= \cos\theta + \frac{\rho}{\alpha^2}\int_0^\infty \lambda(\lambda+\beta)\,(e^{-\beta z} - e^{-\lambda z})\,J_1(\lambda\rho)\,d\lambda, \tag{4.3}$$

where $\rho = r\sin\theta$, $z = r\cos\theta$. The term '$\cos\theta$' satisfies $\mathscr{D}^2\psi^{(0)} = 0$ and the boundary condition $\psi^{(0)} = 0$ when $\theta = \frac{1}{2}\pi$ and is, in fact, the steady solution for a point source in an inviscid fluid.

The integral for $\psi^{(0)}$ can be further expanded in terms of known functions leading to

$$\psi^{(0)}(\rho, z) = \cos\theta - \frac{3\sin^2\theta\cos\theta}{\alpha^2 r^2} - \frac{\rho}{\alpha^2}\int_0^\infty \lambda\beta e^{-\lambda z} J_1(\lambda\rho)\, d\lambda + \frac{\Omega^{(0)}}{\alpha^2},$$

which is a sum of three irrotational terms and a rotational term.

The asymptotic expansion in the far-field of this outer region as $z \to \infty$ can be shown (see appendix) to be

$$\psi^{(0)} = \cos\theta - \frac{\sin^2\theta}{\alpha r} - \frac{3\sin^2\theta\cos\theta}{\alpha^2 r^2} + O(r^{-3}) + \text{exponentially small terms.}$$

The leading term is simply the irrotational source stream function for steady flow, whereas the higher-order terms can be interpreted as a dipole, quadrupole and so on (cf. Tuck). It is in the far part of the outer region that the effects of vorticity can be most easily seen. Reintroducing the real function $\Psi^{(0)}(r, \theta, t)$, its far-field expansion as $z \to \infty$ at 0° phase ($t = 0, 2\pi, \ldots$) is

$$\Psi^{(0)}(r, \theta, 0) \sim \cos\theta - \frac{\sin^2\theta}{2^{\frac{1}{2}} r} + O(r^{-3}),$$

and at 90° phase ($t = \frac{1}{2}\pi, \frac{5}{2}\pi, \ldots$) is

$$\Psi^{(0)}(r, \theta, \tfrac{1}{2}\pi) \sim -\frac{\sin^2\theta}{2^{\frac{1}{2}} r} - \frac{3\sin^2\theta\cos\theta}{r^2} + O(r^{-3}).$$

At 0° phase then, the zeroth-order approximate streamlines in the far field behave as if they are emanating radially from a point source at $\rho = 0$, $z = 1/2^{\frac{1}{2}}$, but at 90° phase the dipole term is dominant and they appear as closed loops in any meridian plane. These results display a very strong analogy to the two-dimensional case as presented by Tuck, and at phases other than 0°, 90°, 180° or 270° it would be expected that there would be some closed streamlines and some radial ones.

The strong analogy between the axisymmetric and the plane two-dimensional case is continued in the near-field expansion of the outer solution. As $r \to 0$ it is shown in the appendix that

$$\psi^{(0)}(r, \theta) \sim \cos^3\theta + \tfrac{1}{4}\alpha^2 r^2 \cos^2\theta(1 - \cos\theta) + O(r^4).$$

Here the leading-order term '$\cos^3\theta$' is a solution of $\mathscr{D}^4\psi^{(0)} = 0$ and indicates that the diffusive terms dominate the time derivative terms in the near-field part of the outer region. The real zeroth-order approximate stream function in this expansion as $r \to 0$ is

$$\Psi^{(0)}(r, \theta, t) = \cos^3\theta\cos t - \tfrac{1}{4}r^2\cos^2\theta(1 - \cos\theta)\sin t + O(r^4)$$

and hence at zero phase, the streamlines in this near field appear to be almost radial, whereas at 90° phase they are approximately the curves given by the solutions of $r^2\cos^2\theta(1 - \cos\theta) = \text{constant}$. These curves resemble hyperbolae in the (ρ, z) plane. The fact that $\Psi^{(0)}(r, \theta, t)$ tends to zero at 90° phase both as r becomes small and r becomes large confirms that there are closed streamlines at this phase.

The leading order term for $\Psi'^{(0)}$ as $r \to 0$ is that for quasi-steady classical viscous flow due to a point source of oscillating strength situated on a wall. This is just the type of flow predicted in §3 for the far part of the inner flow region. Thus the inner and outer unsteady solutions can be matched to zeroth order in an overlapping region, and a uniformly valid composite expansion can be obtained.

5. Conclusion

Oscillatory flow of a viscous incompressible fluid through a circular hole in a wall, such that the volume flux has a constant amplitude, has been analyzed at low Reynolds numbers by dividing the flow field into two regions; an inner one near the hole where the diffusion of vorticity is dominant, and an outer one in which both the diffusive and local rate of change of vorticity are dominant. The flow in the outer region appears to be generated by a three-dimensional point source of oscillating strength. As in the two-dimensional problem (Tuck 1970) the essential feature of the results for this region is the presence of closed streamlines as the singularity switches over from one with a sink-like nature to one with a source-like nature and vice versa.

The inner region problem is quasi-steady in the sense that the results obtained are those for the corresponding steady problem times the harmonic time-dependent term. The important feature of the approximate expression obtained for the stream function in this inner region is that it is self-consistent in the vicinity of the hole. As this feature is also true for the corresponding steady case, the results obtained are therefore useful to a study of suction and injection at low Reynolds numbers.

Solutions with arbitrary time dependence can be constructed from the sinusoidally time-dependent solutions presented in this paper, since the results for both inner and outer regions are obtained by solving linear equations under linear boundary conditions.

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Appendix. Asymptotes of the outer solution

A valid far-field asymptotic expansion with z large is obtained by expanding for small λ under the integral sign. Thus,

$$\begin{aligned}\Omega^{(0)} &= -\rho\int_0^\infty \lambda(\lambda+[\lambda^2+\alpha^2]^{\frac{1}{2}})\exp\left(-[\lambda^2+\alpha^2]^{\frac{1}{2}}z\right)J_1(\lambda\rho)\,d\lambda \\ &= -\rho\int_0^\infty \lambda\left(\alpha+\lambda+\frac{\lambda^2}{2\alpha}+\ldots\right)\exp\left[-(\alpha+(\lambda^2/2\alpha)+\ldots)z\right]J_1(\lambda\rho)\,d\lambda \\ &= -\frac{(2\pi\alpha^5)^{\frac{1}{2}}}{4}\frac{\rho}{z^{\frac{3}{2}}}e^{-\alpha z}.e^{-\rho^2\alpha/4z}\left\{I_0\left(\frac{\rho^2\alpha}{4z}\right)-I_1\left(\frac{\rho^2\alpha}{4z}\right)\right\} \\ &\qquad -\frac{\rho^2}{z^2}e^{-\alpha z}.e^{-\rho^2\alpha/2z}+O(z^{-\frac{5}{2}}e^{-\alpha z}) \quad \text{as} \quad z\to\infty,\end{aligned}$$

which tends to zero like $e^{-\alpha z}$ as $z \to \infty$. Note that the above analysis has assumed nothing about the magnitude of ρ and is, in fact, true for arbitrary finite values of ρ^2/z. When ρ^2/z is also large then it can easily be shown that

$$\Omega^{(0)} \sim -\frac{\alpha\, e^{-\alpha z}}{\rho} + O(z^{-2}\, e^{-\alpha z}) \quad (z \to \infty).$$

Hence it is true that $\Omega^{(0)} \to 0$ exponentially for *all* ρ as $z \to \infty$.

Now it has been shown that

$$\psi^{(0)}(\rho, z) = \cos\theta - \frac{3\sin^2\theta\cos\theta}{\alpha^2 r^2} - \frac{\rho}{\alpha^2}\int_0^\infty \lambda\beta\, e^{-\lambda z} J_1(\lambda\rho)\, d\lambda + \frac{\Omega^{(0)}}{\alpha^2}$$

and the integral in this expression can be expanded as $z \to \infty$ thus:

$$\begin{aligned} &-\frac{\rho}{\alpha^2}\int_0^\infty \lambda[\lambda^2+\alpha^2]^{\frac{1}{2}}\, e^{-\lambda z} J_1(\lambda\rho)\, d\lambda \\ &= -\frac{\rho}{\alpha}\int_0^\infty \lambda\, e^{-\lambda z} J_1(\lambda\rho)\, d\lambda - \frac{\rho}{2\alpha^3}\int_0^\infty \lambda^3\, e^{-\lambda z} J_1(\lambda\rho)\, d\lambda + \ldots \\ &= -\frac{\sin^2\theta}{\alpha r} + \frac{3\sin^4\theta - 12\sin^2\theta\cos^2\theta}{2\alpha^3 r^3} + O(r^{-4}). \end{aligned}$$

This expression is uniformly valid with respect to ρ and hence the asymptotic expansion for $\psi^{(0)}$ for large z is

$$\psi^{(0)} \sim \cos\theta - \frac{\sin^2\theta}{\alpha r} - \frac{3\sin^2\theta\cos\theta}{\alpha^2 r^2} + \frac{3\sin^4\theta - 12\sin^2\theta\cos^2\theta}{2\alpha^3 r^3} + O(r^{-4})$$
$$+\text{exponentially small terms from } \Omega^{(0)}, \text{ as } r \to \infty.$$

This expansion, however, is not valid for large r such that ρ is large and z is small. It can be shown, however, that in this region $\Omega^{(0)}$ tends to zero but the decay is *not* exponential.

Similarly, the asymptotic expansion for small r can be obtained, at least for the first few terms, by expanding for large λ and keeping $\lambda\rho$ and λz bounded.

$$\begin{aligned} \psi^{(0)}(\rho, z) &= \cos\theta + \frac{\rho}{\alpha^2}\int_0^\infty \lambda(\lambda+\beta)\,(e^{-\beta z} - e^{-\lambda z})\, J_1(\lambda\rho)\, d\lambda \\ &= \cos\theta + \frac{\rho}{\alpha^2}\int_0^\infty \lambda\left(2\lambda + \frac{\alpha^2}{2\lambda} + \ldots\right) e^{-\lambda z}\,(\exp[-\alpha^2 z/2\lambda + \ldots] - 1)\, J_1(\lambda\rho)\, d\lambda \\ &= \cos\theta - \sin^2\theta\cos\theta + \tfrac{1}{4}\alpha^2 r^2\cos^2\theta(1-\cos\theta) + O(r^4) \\ &= \cos^3\theta + \tfrac{1}{4}\alpha^2 r^2\cos^2\theta(1-\cos\theta) + O(r^4) \quad \text{as} \quad r \to 0. \end{aligned}$$

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